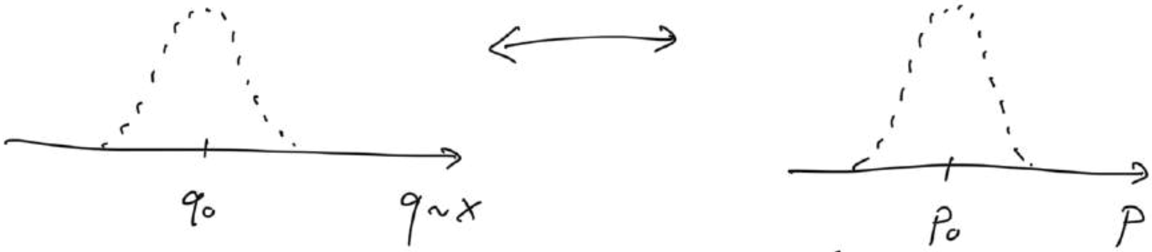
Rozplývání vlnového balíku



$$\Psi(x,t) = N\int_{-\infty}^{\infty} dp\, e^{-\frac{(p-p_0)^2}{\Delta^2}} \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar}(px-Et)}$$

Normalizace

$$1 = \int_{-\infty}^{\infty} dx\, \Psi^*(x)\,\Psi(x) = N^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp'\, e^{-\frac{(p-p_0)^2}{\Delta^2}}\, e^{-\frac{(p'-p_0)^2}{\Delta^2}}$$

$$\times \frac{1}{2\pi\hbar}\, e^{\frac{i}{\hbar}(p-p')x}\, e^{-\frac{i}{\hbar}(E-E')t}$$

$$= N^2 \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp'\, e^{-\frac{(p-p_0)^2}{\Delta^2}-\frac{(p'-p_0)^2}{\Delta^2}}\, \delta(p-p')\, e^{-\frac{i}{\hbar}(E-E')t}$$

$$= N^2 \int_{-\infty}^{\infty} dp\, e^{-\frac{2(p-p_0)^2}{\Delta^2}} = N^2\sqrt{\frac{\pi\Delta^2}{2}} = 1$$

$$N = \left(\frac{2}{\pi\Delta^2}\right)^{1/4}$$

Normalizace

$$\boxed{\Psi(x,t) = \left(\frac{2}{\pi\Delta^2}\right)^{1/4} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, e^{-\frac{(p-p_0)^2}{\Delta^2}}\, e^{\frac{i}{\hbar}(px - Et)}}$$

Střední hodnota impulsu

$$\langle \hat{p} \rangle = -i\hbar \int_{-\infty}^{\infty} dx\, \Psi^*(x,t) \frac{\partial}{\partial x} \Psi(x,t) = ?$$

$$-i\hbar \frac{\partial}{\partial x} \Psi(x,t) = \left(\frac{2}{\pi\Delta^2}\right)^{1/4} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\, p\, e^{-\frac{(p-p_0)^2}{\Delta^2}}\, e^{\frac{i}{\hbar}(px - Et)}$$

$$\boxed{\langle \hat{p} \rangle =} \left(\frac{2}{\pi\Delta^2}\right)^{1/2} \underbrace{\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp' \int_{-\infty}^{\infty} dp\, p\, e^{-\frac{(p-p_0)^2}{\Delta^2}}\, e^{-\frac{(p'-p_0)^2}{\Delta^2}} \int_{-\infty}^{\infty} dx\, e^{\frac{i}{\hbar}(p-p')x}}_{\delta(p-p')} \times e^{-\frac{i}{\hbar}(E(p)-E(p'))t}$$

$$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i(\omega-\omega')t}$$

$$= \left(\frac{2}{\pi\Delta^2}\right)^{1/2} \int_{-\infty}^{\infty} dp\, e^{-\frac{2(p-p_0)^2}{\Delta^2}}\, p = \quad \xrightarrow{\text{subst.}} \; p' = p - p_0$$

$$= \left(\frac{2}{\pi\Delta^2}\right)^{1/2} \int_{-\infty}^{\infty} dp'\, \underline{e^{-\frac{2p'^2}{\Delta^2}}}\, (p' + p_0) = \boxed{p_0}$$

sudá na $\mathbb{R}$

p' – lichá na $\mathbb{R}$

Střední hodnota kvadrátu $\hat{p}$

$\langle \hat{p}^2 \rangle = ?$

$$-\hbar^2 \frac{\partial^2}{\partial x^2} \Psi(x,t) = \dots \text{ stejnou metodou}$$

$$= \left(\frac{2}{\pi \Delta^2}\right)^{1/4} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp \; p^2 e^{-\frac{(p-p_0)^2}{\Delta^2}} e^{\frac{i}{\hbar}(px - Et)}$$

$$\langle \hat{p}^2 \rangle = \left(\frac{2}{\pi \Delta^2}\right)^{1/2} \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp' \int_{-\infty}^{\infty} dp \; p^2 e^{-\frac{(p-p_0)^2}{\Delta^2}} e^{-\frac{(p'-p_0)^2}{\Delta^2}}$$

$$\times \underbrace{\int_{-\infty}^{\infty} dx \, e^{\frac{i}{\hbar}(p-p')x}}_{= \delta(p-p')} e^{-\frac{i}{\hbar}(E-E')t}$$

$$= \left(\frac{2}{\pi \Delta^2}\right)^{1/2} \int_{-\infty}^{\infty} dp \; p^2 e^{-\frac{2(p-p_0)^2}{\Delta^2}}$$

substituce
$p - p_0 = p'$
$p = p' + p_0$

$$= \left(\frac{2}{\pi \Delta^2}\right)^{1/2} \int_{-\infty}^{\infty} dp' (p'+p_0)^2 e^{-\frac{2p'^2}{\Delta^2}}$$

$$= \left(\frac{2}{\pi \Delta^2}\right)^{1/2} \left[\int_{-\infty}^{\infty} dp' p'^2 + \underbrace{2p_0 \int_{-\infty}^{\infty} dp' p' e^{-\frac{2p'^2}{\Delta^2}}}_{=0} + p_0^2 \left(\frac{\pi \Delta^2}{2}\right)^{1/2} \right]$$

$$= p_0^2 + \left(\frac{2}{\pi\Delta^2}\right)^{1/2} \lim_{a\to 1} \frac{\partial}{\partial a} \int_{-\infty}^{\infty} dp'\, e^{-\frac{2p'^2}{\Delta^2} a} \cdot \frac{\Delta^2}{2}$$

$$= p_0^2 + \left(\frac{2}{\pi\Delta^2}\right)^{1/2} \lim_{a\to 1} \frac{\partial}{\partial a} \sqrt{\frac{\pi\Delta^2}{2a}} \left(-\frac{\Delta^2}{2}\right)$$

$$= p_0^2 + \left(-\frac{\Delta^2}{2}\right) \lim_{a\to 1} \frac{\partial}{\partial a} \frac{1}{\sqrt{a}} \qquad \frac{\partial}{\partial a} a^{-1/2} = -\frac{1}{2} a^{-3/2}$$

$$= p_0^2 + \frac{\Delta^2}{4}$$

$$\langle p^2 \rangle = p_0^2 + \frac{\Delta^2}{4} \Rightarrow (\Delta p)^2 = \frac{\Delta^2}{4}$$

$$\boxed{\Delta p = \frac{\Delta}{2}}$$

Stav částice v $t=0$

$$\Psi(x, t=0) \sim \int_{-\infty}^{\infty} dp\, e^{-\frac{(p-p_0)^2}{\Delta^2}} e^{\frac{i}{\hbar} p x} =$$

$$= \int_{-\infty}^{\infty} dp'\, e^{-\frac{p'^2}{\Delta^2}} e^{\frac{i}{\hbar}(p'+p_0)x} \quad \Leftarrow \quad p' = p - p_0, \quad dp' = dp$$

$$= e^{\frac{i}{\hbar} p_0 x} \int_{-\infty}^{\infty} dp'\, e^{-\frac{p'^2}{\Delta^2}} e^{\frac{i}{\hbar} p' x} \qquad \xi = \frac{p'}{\hbar} \quad p' = \hbar \xi$$

$$dk = \frac{1}{\hbar} dp'$$
$$dp' = \hbar\, dk$$

$$= e^{\frac{i}{\hbar} p_0 x} \hbar \int_{-\infty}^{\infty} dk\, e^{-\frac{\hbar^2}{\Delta^2} k^2} e^{ikx} = e^{\frac{i}{\hbar} p_0 x} \hbar \sqrt{\frac{\pi \Delta^2}{\hbar^2}} e^{-\frac{x^2 \Delta^2}{4\hbar^2}}$$

$$= e^{\frac{i}{\hbar} p_0 x} \sqrt{\pi}\, \Delta\, e^{-\frac{\Delta^2}{4\hbar^2} x^2}$$

$$\boxed{\Psi(x,0)} = \left(\frac{2}{\pi \Delta^2}\right)^{1/4} \frac{\Delta}{\sqrt{2\hbar}} e^{\frac{i}{\hbar} p_0 x} e^{-\frac{\Delta^2}{4\hbar^2} x^2}$$

$$= \boxed{\left(\frac{2}{\pi}\right)^{1/4} \left(\frac{\Delta}{2\hbar}\right)^{1/2} e^{\frac{i}{\hbar} p_0 x} e^{-\frac{\Delta^2}{4\hbar^2} x^2}}$$

Střední hodnota $\hat{x}^2$

$$\langle \hat{x}^2 \rangle = \int_{-\infty}^{\infty} dx\, \frac{1}{\sqrt{2\pi}} \left(\frac{\Delta}{\hbar}\right) e^{-\frac{\Delta^2}{2\hbar^2} x^2} x^2 =$$

$$= \frac{\Delta}{\sqrt{2\pi}\,\hbar} \left(-\frac{2\hbar^2}{\Delta^2}\right) \lim_{a \to 1} \frac{\partial}{\partial a} \int_{-\infty}^{\infty} dx\, e^{-\frac{\Delta^2}{2\hbar^2} x^2 a}$$

$$= \frac{\Delta}{\sqrt{2\pi}\,\hbar} \left(-\frac{2\hbar^2}{\Delta^2}\right) \lim_{a \to 1} \frac{\partial}{\partial a} \sqrt{\frac{\pi\, 2\hbar^2}{a \Delta^2}}$$

$$= \frac{\Delta}{\sqrt{2\pi}\,\hbar} \left(-\frac{2\hbar^2}{\Delta^2}\right) \sqrt{2\pi}\, \frac{\hbar}{\Delta} \lim_{a \to 1} \frac{\partial}{\partial a} a^{-1/2}$$

$$= -\frac{2\hbar^2}{\Delta^2}\lim_{a\to 1}\left(-\frac{1}{2}\right)a^{-3/2}$$

$$= \frac{\hbar^2}{\Delta^2}$$

$$\boxed{\langle \hat{p}^2\rangle = \frac{\hbar^2}{\Delta^2}}$$

Časový vývoj:

$\Psi(x,t)$ splňuje Schrödingerovu rovnici ; $E = \frac{p^2}{2m}$

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \int_{-\infty}^{\infty} dp \left(\frac{2}{\pi\Delta^2}\right)^{1/4} e^{-\frac{(p-p_0)^2}{\Delta^2}} \frac{1}{\sqrt{2\pi\hbar}}\frac{i}{\hbar}\left(-\frac{p^2}{2m}\right) i\hbar\, e^{\frac{i}{\hbar}(px-Et)}$$

$$\parallel$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) = \int_{-\infty}^{\infty} dp \left(\frac{2}{\pi\Delta^2}\right)^{1/4} e^{-\frac{(p-p_0)^2}{\Delta^2}} \frac{1}{\sqrt{2\pi\hbar}}\left(-\frac{\hbar^2}{2m}\right)\left(\frac{i}{\hbar}\right)^2 p^2 e^{\frac{i}{\hbar}(px-Et)}$$

Časové odvíjení vlnového balíku

$$\Psi(x,t=0) = \left(\frac{2}{\pi}\right)^{1/4}\left(\frac{\Delta}{2\hbar}\right)^{1/2} e^{\frac{i}{\hbar}p_0 x}\, e^{-\frac{\Delta^2}{4\hbar^2}x^2}$$

$$= N_0\, e^{-\frac{\Delta^2}{4\hbar^2}\left(\underbrace{x^2}_{A^2} - \underbrace{i\frac{4\hbar^2}{\hbar\Delta^2}p_0 x}_{2BA} - \underbrace{\frac{4\hbar^2 p_0^2}{\Delta^4}}_{B^2}\right) - \cancel{\frac{\Delta^2}{4\hbar^2}}\frac{\cancel{4\hbar^2}p_0^2}{\Delta^4}}$$

$$B = \frac{2\hbar p_0}{\Delta^2}$$

$$= N_0\, e^{-\frac{p_0^2}{\Delta^2}}\, e^{-\frac{\Delta^2}{4\hbar^2}\left(x - i\frac{2\hbar p_0}{\Delta^2}\right)^2}$$

Funkce rozšířená o správnou časovou závislost $\Rightarrow$

$$\psi(x,t) = \frac{N_0}{\sqrt{1+2iat}}\, e^{-\frac{p_0^2}{\Delta^2}}\, e^{-\frac{\Delta^2}{4\hbar^2}\frac{\left(x - i\frac{2\hbar p_0}{\Delta^2}\right)^2}{1+2iat}}$$

$$\frac{\partial}{\partial t}\psi(x,t) = N_0 \frac{\partial}{\partial t}(1+2iat)^{-1/2} e^{\cdots} + N_0 (1+2iat)^{-1/2}$$

$$\times \frac{\partial}{\partial t}\left(\frac{-\frac{\Delta^2}{4\hbar^2}\left(x - i\frac{2\hbar p_0}{\Delta^2}\right)^2}{1+2iat}\right) e^{\cdots}$$

$$= N_0\left(-\frac{1}{2}\, 2ia\right)(1+2iat)^{-3/2} e^{\cdots} + N_0 (1+2iat)^{-1/2}(2ia)$$

$$\times\left(-\frac{\Delta^2}{4\hbar^2}\left(x - i\frac{2\hbar p_0}{\Delta^2}\right)^2\right)(-1)(1+2ia)^{-2}$$

$$\times e^{\cdots}$$

$$= -ia(1+2iat)^{-1}\psi(x,t) + 2ia\frac{\Delta^2}{4\hbar^2}\left(x - i\frac{2\hbar p_0}{\Delta^2}\right)^2$$

$$\times (1+2iat)^{-2}\psi(x,t)$$

$$\frac{\partial}{\partial x}\psi(x,t) = \frac{-2\frac{\Delta^2}{4\hbar^2}\left(x - i\frac{2\hbar p_0}{\Delta^2}\right)}{1+2iat}\psi(x,t)$$

$$\frac{\partial^2}{\partial x^2}\psi(x,t) = -\frac{\Delta^2}{2\hbar^2}(1+2iat)^{-1}\psi(x,t) - \frac{\Delta}{2\hbar^2}\frac{\left(x - i\frac{2\hbar p_0}{\Delta^2}\right)}{1+2iat}$$

$$\times\left(-\frac{\Delta^2}{4\hbar^2}\,2\,\frac{\left(x-i\cdot\frac{2\hbar p_0}{\Delta^2}\right)}{1+2iat}\right)\psi(x,t)$$

$$=-\frac{\Delta^2}{2\hbar^2}(1+2iat)^{-1}\psi(x,t)+\left(\frac{\Delta^2}{2\hbar^2}\right)^2\frac{\left(x-i\cdot\frac{2\hbar p_0}{\Delta^2}\right)^2}{(1+2iat)^2}\psi(x,t)$$

$$\Rightarrow\ \frac{\partial}{\partial t}\psi(x,t)=-ia\left[\frac{\psi(x,t)}{(1+2iat)}-\frac{\Delta^2}{2\hbar^2}\frac{\left(x-i\frac{2\hbar p_0}{\Delta^2}\right)^2}{(1+2iat)^2}\psi(x,t)\right]$$

$$\frac{\partial^2}{\partial x^2}\psi(x,t)=-\frac{\Delta^2}{2\hbar^2}\left[\frac{\psi(x,t)}{(1+2iat)}-\frac{\Delta^2}{2\hbar^2}\frac{\left(x-i\frac{2\hbar p_0}{\Delta^2}\right)^2}{(1+2iat)^2}\psi(x,t)\right]$$

schrödingerova rovnice

$$i\frac{\partial}{\partial t}\psi(x,t)=-\frac{\hbar^2}{\cancel{\hbar}2m}\frac{\partial^2}{\partial x^2}\psi(x,t)$$

$$a\left[\ \dots\ \right]=-\frac{\hbar}{2m}\left(-\frac{\Delta^2}{2\hbar^2}\right)\left[\ \dots\ \right]$$

$$\Downarrow$$

$$a=\frac{\Delta^2}{4m\hbar}$$

$$\boxed{\psi(x,t)=\frac{\left(\frac{2}{\pi}\right)^{1/4}\left(\frac{\Delta}{2\hbar}\right)^{1/2}}{\sqrt{1+2i\frac{\Delta^2}{4m\hbar}t}}\,e^{-\frac{p_0^2}{\Delta^2}}\,e^{-\frac{\Delta^2}{4\hbar^2}\frac{\left(x-i\cdot\frac{2\hbar p_0}{\Delta^2}\right)^2}{1+i\frac{\Delta^2}{2m\hbar}t}}}$$

$$a=\frac{\Delta^2}{2m\hbar^2}\qquad i\beta=\left(\frac{\Delta}{2\hbar}\right)^2\qquad \xi_0=\frac{2\hbar p_0}{\Delta^2}$$

$$\left(\frac{p_0}{\Delta}\right)^2 = \beta \xi_0^2$$

$$\psi(x,t) = \frac{N_0}{\sqrt{1+iat}}\, e^{-\beta\xi_0^2}\, e^{-\frac{\beta(x-i\xi_0)^2}{1+iat}}$$

$$= \frac{N_0}{\sqrt{1+iat}}\, e^{-\beta\frac{\left[(x-i\xi_0)^2+\xi_0^2(1+iat)\right]}{1+iat}}$$

$$= \frac{N_0}{\sqrt{1+iat}}\, e^{-\beta\frac{\left[x^2-2i\xi_0 x-\xi_0^2+\xi_0^2+i\xi_0^2 at\right]}{1+iat}}$$

$$= \frac{N_0}{\sqrt{1+iat}}\, e^{-\beta\frac{\left[x^2-i(2\xi_0 x-\xi_0^2 at)\right]}{1+iat}}$$

$$= \frac{N_0}{\sqrt{1+iat}}\, e^{-\beta\frac{\left[x^2-i(2\xi_0 x-\xi_0 at)\right]}{1+a^2t^2}(1-iat)}$$

$$= \frac{N_0}{\sqrt{1+iat}}\, e^{-\beta\frac{\left[x^2-i(2\xi_0 x-\xi_0^2 at)-iatx^2-(2\xi_0 x-\xi_0^2 at)at\right]}{1+a^2t^2}}$$

$$= \frac{N_0}{\sqrt{1+iat}}\, e^{-\beta\frac{\left[x^2-2a\xi_0 xt+\xi_0^2a^2t^2\right]}{1+a^2t^2}+i\beta\frac{[\dots]}{1+a^2t^2}}$$

poloha centra

$$= \frac{N_0}{\sqrt{1+iat}}\, e^{-\beta\frac{\left[x-\xi_0 at\right]^2}{1+a^2t^2}}\, e^{i\beta\left[(2\xi_0-atx)x-\xi_0^2 at\right]}$$

šířka

fáze

$$= \frac{N_0}{\sqrt{1+iat}}\, e^{-\beta\frac{\left[x-\xi_0 at\right]^2}{1+a^2t^2}}\, e^{i\beta\left[(2\xi_0-atx)x-\xi_0^2 at\right]}$$

$\sim$(šířka balíku)^2

$$(\text{šířka})^2 \sim \frac{1}{\beta} + \frac{a^2}{\beta} t^2 = \left(\frac{2\hbar}{\Delta}\right)^2 + \frac{\Delta^2}{m^2} t^2$$

$$= \delta^2 + \left(\frac{1}{\delta}\right)^2 \frac{4\hbar^2}{m^2} t^2$$

pro $\hbar \to 0$, $m \to \infty$ } zpomalujte rozplývání

čím menší počáteční lokalizace, tím rychlejší rozšiřování

Volná částice v obecném stavu

gaussovský balík

$$\langle \hat{p} \rangle = p_0$$

$$\langle \hat{p}^2 \rangle = p_0^2 + \frac{\Delta^2}{4}$$

$$\langle \hat{x} \rangle = 0$$

$$\langle \hat{x}^2 \rangle = \frac{\hbar^2}{\Delta^2}$$

$$(\Delta p)^2 = \frac{\Delta^2}{4}$$

$$(\Delta x)^2 = \frac{\hbar^2}{\Delta^2}$$

$$\boxed{\Delta p \cdot \Delta x = \frac{\Delta}{2} \frac{\hbar}{\Delta} = \frac{\hbar}{2}}$$

časový vývoj:

$$\frac{d}{dt} \langle \hat{p}(t) \rangle = \left\langle \frac{i}{\hbar} [\hat{H}, \hat{p}] \right\rangle = 0$$

$$\frac{d}{dt} \langle \hat{p}^2(t) \rangle = 0$$

$$\hat{p}(t) = \hat{p}_0$$

$$\frac{d\hat{p}(t)}{dt} = 0$$

$$\boxed{\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar}[\hat{H},\hat{x}(t)] = \frac{i}{\hbar}U^{+}(t)[\hat{H},\hat{x}]U(t) = \frac{\hat{p}}{m}}$$

$$\frac{\hat{p}^2}{2m}\hat{x} - \hat{x}\frac{\hat{p}^2}{2m} = -\frac{i\hbar}{m}\hat{p}$$

pro volnou částici $\hat{p}$ a $\hat{U}(t)$ komutují

$$\Rightarrow \boxed{\hat{x}(t) = \frac{\hat{p}}{m}t + \hat{x}}$$

$$\langle \hat{x}^2(t)\rangle = \left\langle \left(\frac{\hat{p}t}{m} + \hat{x}\right)\left(\frac{\hat{p}t}{m} + \hat{x}\right)\right\rangle$$

$$= \frac{1}{m^2}\langle \hat{p}^2\rangle t^2 + \langle \hat{x}^2\rangle + \langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle \frac{t}{m}$$

gaussovská částice $\rightarrow$ $\langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle = 0$

$$\langle (\hat{x}(t))^2\rangle = \frac{\left(p_0^2 + \frac{\Delta^2}{4}\right)}{m^2}t^2 + \frac{\hbar^2}{\Delta^2} = \left(\frac{p_0 t}{m}\right)^2 + \frac{\Delta^2}{4m^2}t^2 + \frac{\hbar^2}{\Delta^2}$$

pro $p_0 = 0$

$$\boxed{(\Delta x(t))^2 (\Delta p(t))^2} = \left(\frac{\Delta^2}{4m^2}t^2 + \frac{\hbar^2}{\Delta^2}\right)\frac{\Delta^2}{4} =$$

$$\boxed{= \frac{\hbar^2}{4} + \frac{\Delta^4}{16m^2}t^2}$$