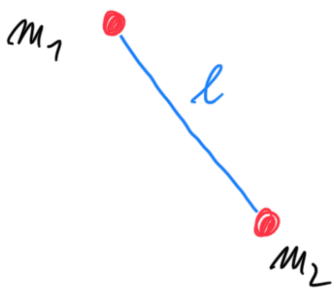
Tuhý rotátor



$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$T = \frac{1}{2} m v^2 = \frac{p^2}{2\mu}$

$p = m v$

Kinetická energie

$$T = \frac{1}{2} I \omega^2$$

$$L = I\omega \Rightarrow T = \frac{L^2}{2I}$$

$$I = \mu l^2$$

$$\hat{H} = \hat{T} + \hat{V} \quad ; \hat{V} = 0$$

$$= -\frac{\hbar^2}{2\mu} \nabla^2 \ \ldots \ \text{očekáváme} \ \frac{L^2}{2I}$$

$$\hat{H} = -\frac{\hbar^2}{2\mu} \nabla^2 = \frac{\hat{L}^2}{2I} = \frac{\hat{L}^2}{2\mu l^2} \Rightarrow \hat{L}^2 = -\hbar^2 l^2 \nabla^2 \Big|_{r=l}$$

Využijeme:

$\hat{L}_z, \hat{L}^2, \hat{H}$ spolu komutují $\Rightarrow$ mají společné vlastní vektory

vlastní problém pro $\hat{L}_z$

$$L_z = -i\hbar \frac{\partial}{\partial \varphi}$$

$$\hat{L}_z \Psi(\varphi) = l_z \Psi(\varphi)$$

platí, že $\Psi(\varphi + 2\pi) = \Psi(\varphi)$

$$\Psi(\varphi) = A e^{i\varphi m}$$

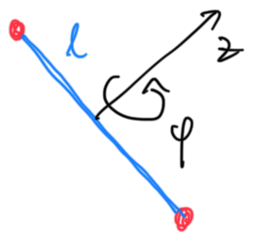
$$\Psi(\varphi+2\pi) = A e^{i\varphi m} \underbrace{e^{i2\pi m}}_{1} = \Psi(\varphi)$$

$$L_z \Psi(\varphi) = -i\hbar\,(im)\,A e^{im\varphi} = \underline{m\hbar}\,\Psi(\varphi)$$

Normalizace
$$\int_0^{2\pi} d\varphi\, A^2 e^{-im\varphi} e^{im\varphi} = 2\pi A^2 = 1 \Rightarrow A = \frac{1}{\sqrt{2\pi}}$$

$$\Psi_m(\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}} \qquad L_z|m\rangle = m|m\rangle$$

v jednotkách $\hbar$



$$\hat L_\pm = \hat L_x \pm i\hat L_y$$

$$L_zL_x - L_xL_z = [L_z, L_x]$$
$$L_zL_x = [L_z, L_x] + L_xL_z$$

$$L_z L_\pm |m\rangle = L_z(L_x \pm iL_y)|m\rangle$$
$$= \left([L_z, L_x] \pm i[L_z, L_y] + (L_x \pm iL_y)L_z\right)|m\rangle$$

$iL_y \qquad -iL_x$

$$= \left(iL_y \pm L_x + L_\pm L_z\right)|m\rangle \overset{①+}{=} \left(L_x + iL_y + L_+ m\right)|m\rangle$$

②−
$$= \Big(\underbrace{-L_x + iL_y}_{\substack{-(L_x - iL_y)\\ L_-}} + L_- m\Big)|m\rangle$$

$$\overset{+}{=} (m+1)\,L_+|m\rangle$$

$$\ominus$$

$$= (m-1)\, L_- |m\rangle$$

L_+ zvyšovací operátor

L_- snižovací operátor

kvantové číslo průmětu složky momentu hybnosti do osy z.

Vlastní stavy $\hat{L}^2$

vlastní číslo

$$\hat{L}^2 |j\rangle = (\mathcal{L}^2)_j |j\rangle$$

$$\hat{L}^2 \underbrace{L_\pm |j\rangle}_{|\pm j\rangle} = \hat{L}^2 (L_x \pm L_y)|j\rangle = \left([\hat{L}^2, L_x] + L_x \hat{L}^2 \pm i [\hat{L}^2, L_y] \pm i L_y \hat{L}^2\right) |j\rangle$$

$$= \left([\hat{L}^2, L_x] \pm i [\hat{L}^2, L_y] + (L_x \pm i L_y)\hat{L}^2\right)|j\rangle$$

$$= \mathcal{L}_j^2 |\pm j\rangle + [\hat{L}^2, L_x]|j\rangle \pm i [\hat{L}^2, L_y]|j\rangle$$

$$[\hat{L}^2, L_x] \pm i [\hat{L}^2, L_y]$$

$$= [\hat{L}^2, L_x \pm i L_y] = 0$$

$$\hat{L}^2 |\pm j\rangle = (\mathcal{L}_j^2) |\pm j\rangle$$

vlastní stav $\hat{L}^2$

$$|jm\rangle$$

$$L_\pm |jm\rangle = \,?$$

$$\boxed{\hat{L}^2 = L_+ L_- + L_z^2 - L_z}$$

$$L_+ L_- = (L_x + iL_y)(L_x - iL_y) =$$
$$= L_x^2 + L_y^2 + i\underbrace{(L_y L_x - L_x L_y)}_{[L_y, L_x]} =$$
$$= L_x^2 + L_y^2 + L_z$$
$$\boxed{= \hat{L}^2 + L_z - L_z^2}$$

$$L_+ L_- |jm\rangle = (L^2 - L_z^2 + L_z)|jm\rangle = \left((\alpha^2)_j - m^2 + m\right)|jm\rangle$$
$$= \left[(\alpha^2_j) - m(m-1)\right]|jm\rangle$$

$$\boxed{L_\pm |jm\rangle = \alpha^\pm(j,m)\,|jm\pm 1\rangle}$$

$$(\hat{L}_+)^\dagger = (\hat{L}_x)^\dagger + i(\hat{L}_y)^\dagger = \hat{L}_x - i\hat{L}_y = \hat{L}_-$$

$$\langle jm|L_+ L_-|jm\rangle = \left[(\alpha^2)_j - m(m-1)\right] \equiv |\alpha^-(j,m)|^2$$

$$\Rightarrow \alpha^-(j,m) = \sqrt{(\alpha^2)_j - m(m-1)}$$

$$\langle jm|L_+|j,m-1\rangle = \underline{\alpha^+(j,m-1)}$$
$$= (L_-|jm\rangle)^\dagger |jm-1\rangle = \underline{\alpha^-(j,m)}\langle jm-1|jm-1\rangle$$

$$\alpha^+(j,m-1) = \alpha^-(j,m)$$

$$\boxed{\alpha^-(j,m) = \alpha^+(j,m-1) = \sqrt{(\alpha^2)_j - m(m-1)}}$$

Musí existovat m_{max} a m_{min}

$$L_-|j\,m_{min}\rangle = 0$$

$$L_-|j\,m_{min}\rangle = \underbrace{\alpha^-(j,m_{min})}_{=0}|j,m_{min-1}\rangle = 0$$

$$\sqrt{(\alpha^2)_j - m_{min}(m_{min}-1)} = 0 \Rightarrow (\alpha^2)_j = m_{min}(m_{min}-1) =$$

$m_{min} < 0$

$$= -|m_{min}|(-|m_{min}|-1)$$

$$= |m_{min}|(|m_{min}|+1)$$

$$L_+|j\,m_{max}\rangle = \alpha^+(j,m_{max})|j\,m_{max}+1\rangle = 0$$

$$\alpha^+(j,m_{max}) = \sqrt{(\alpha^2)_j - m_{max}(m_{max}+1)}$$

$m_{max} > 0$

$$\boxed{(\alpha^2)_{j_{max}} = j(j+1)}$$

$$j \equiv |m_{min}| = |m_{max}|$$

Zpět k tuhému rotátoru

chybí ve videu

$$\hat{H} = \frac{\hat{L}^2}{2\mu l} = \frac{\hat{L}^2}{2I}\,; \quad \hat{H}|jm\rangle = \frac{\hbar^2 j(j+1)}{2I}|jm\rangle$$

$$m = -j \ldots, j \Rightarrow 2j+1$$

možných stavů

Vlastní funkce kvadrátu momentu hybnosti

$$\hat{L}^2 \underbrace{Y_{lm}(\vartheta,\varphi)}_{\langle \vartheta,\varphi | lm\rangle} = l(l+1)\, Y_{lm}(\vartheta,\varphi)$$

$$\hat{L}_z Y_{lm}(\vartheta,\varphi) = m\, Y_{lm}(\vartheta,\varphi)$$

$$Y_{lm}(\vartheta,\varphi) = P_{lm}(\vartheta)\, e^{im\varphi}$$ m ... celé číslo

$$l = |m_{max}| = |m_{min}|$$

Zde, stačí pouze $l = 1$ $m = -1, 0, +1$

$$\hat{L}_- |1,-1\rangle = 0 \quad \Rightarrow \quad \hat{L}_- Y_{1,-1}(\vartheta,\varphi) = 0$$

Vyjádřit $\hat{L}_x$ a $\hat{L}_y$ ve sférických souřadnicích $\Rightarrow \hat{L}_- = \hat{L}_x - i\hat{L}_y$

$$\hat{L}_\pm = e^{\pm i\varphi}\left(\pm \frac{\partial}{\partial\vartheta} + i \cot\vartheta \frac{\partial}{\partial\varphi}\right)$$

ve videu chybí ϑ

$$\hat{L}_- Y_{1,-1}(\vartheta,\varphi) = e^{-i\varphi}\left(-\frac{\partial}{\partial\vartheta} + i\cot\vartheta \frac{\partial}{\partial\varphi}\right) P_{1,-1}(\vartheta)\, e^{-i\varphi}$$

$$= e^{-i\varphi}\left(-\frac{\partial}{\partial\vartheta} P(\vartheta)\, e^{-i\varphi}\right) + \cot\vartheta\, P_{1,-1}(\vartheta)\, e^{-i2\varphi} = 0$$

$$\frac{\partial}{\partial\vartheta} P_{1,-1}(\vartheta) = \frac{\cos\vartheta}{\sin\vartheta} P_{1,-1}(\vartheta) \Rightarrow P_{-1,1}(\vartheta) = K \sin\vartheta$$

Normalizace integrací

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin\vartheta \; K^2 \sin^2\vartheta = 1$$

$$2\pi K^2 \int_0^{\pi} d\vartheta \sin^3\vartheta = 1$$

$$\sin^3\vartheta = -\frac{1}{4}\sin 3\vartheta + \frac{3}{4}\sin\vartheta$$

$$\int_0^{\pi} d\vartheta \sin\vartheta = 2$$

$$\int_0^{\pi} d\vartheta \sin 3\vartheta = \frac{2}{3}$$

$$\frac{8}{3}\pi K^2 = 1 \Rightarrow K = \sqrt{\frac{3}{8\pi}}$$

$$\boxed{Y_{1,-1}(\vartheta,\varphi) = \sqrt{\frac{3}{8\pi}} \sin\vartheta \, e^{-i\varphi}}$$

Operator L_+:

$$L_+ \underset{\ell\ m}{Y_{1,-1}} = \alpha^+\underset{\ell\ m}{(1,-1)}\, Y_{1,0}$$

$$\alpha^+(\ell,m) = \sqrt{\ell(\ell+1) - m(m+1)}$$

$$\alpha^+(1,-1) = \sqrt{2}$$

$$Y_{1,0}(\vartheta,\varphi) = \frac{1}{\sqrt{2}} L_+ Y_{1,-1}(\vartheta,\varphi) \qquad \alpha^+(1,0) = \sqrt{2}$$

$$Y_{1,1}(\vartheta,\varphi) = \frac{1}{\sqrt{2}} L_+ Y_{1,0}(\vartheta,\varphi)$$

Dostaneme:

$$\boxed{\begin{aligned} Y_{10}(\vartheta,\varphi) &= \sqrt{\frac{3}{4\pi}}\cos\vartheta \\ Y_{11}(\vartheta,\varphi) &= -\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{i\varphi} \end{aligned}}$$